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Free resolutions and Koszul homology

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Abstract

Let $(R, \mathfrak{m})$ be a regular local ring, and M an R-module. The minimal free resolution F of M has a natural $\mathfrak{m}$-adic filtration. In this paper we establish an isomorphism of the spectral sequence which is associated to the filtered complex F, and the spectral sequence associated to a double complex derived from the Koszul complex of M with respect to a minimal system of generators of $\mathfrak{m}$. We use this result to describe explicitly the maps in the resolution of terms of Koszul cycles whenever the resolution of the module is pure.

0. Introduction

Let $(R, \mathfrak{m}, k)$ be a regular local ring, $\boldsymbol{x} = x_1, \ldots, x_n$ a regular system of parameters of R, and M a finitely generated R-module. In this paper we want to show that the minimal free resolution (G, d) of M can be partially recovered from the cycles generating the Koszul homology $H(\boldsymbol{x}; M)$.

The opposite problem, namely to compute the Koszul cycles from the resolution, has been treated in [5] by the second author for graded modules over a polynomial ring by making explicit the natural isomorphism $k \otimes_R G \cong H(\boldsymbol{x}; M)$. Note that by this isomorphism the rank of the free module G_q in the resolution G is just $\dim_k H(\boldsymbol{x}; M)$. Thus it remains to construct the differential d of G from the cycles of the Koszul complex $K(\boldsymbol{x}; M)$.

In case M has a linear resolution this problem has already been solved in the paper [7] or [3]. One considers the (ascending) filtration F on G with $F_p G_q = \mathfrak{m}^{-(p+q)} G_q$ for all p and q; here $\mathfrak{m}^s G_q = G_q$ if $s \le 0$. The resolution of M is called linear if the associated graded complex $\operatorname{gr}_F(G)$ is acyclic. In this case, $\operatorname{gr}_F(G)$ is the minimal graded

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free resolution of the graded $\mathrm{gr}_{\mathfrak{m}}(R)$-module $\mathrm{gr}_{\mathfrak{m}}(M)$ and $\mathrm{gr}_F(G)$ is linear in the sense that the differential $\mathrm{gr}_F(d)$ is described by matrices of linear forms. In [7, Theorem 5.1] it is shown that $\mathrm{gr}_F(G)$ is isomorphic to the complex $H(\boldsymbol{x}; M) \otimes_k \mathrm{gr}_{\mathfrak{m}}(R)$:

$$\cdots \xrightarrow{\vartheta} H_2(\boldsymbol{x}; M) \otimes_k \mathrm{gr}_{\mathfrak{m}}(R) \xrightarrow{\vartheta} H_1(\boldsymbol{x}; M) \otimes_k \mathrm{gr}_{\mathfrak{m}}(R) \xrightarrow{\vartheta} H_0(\boldsymbol{x}; M) \otimes_k \mathrm{gr}_{\mathfrak{m}}(R) \longrightarrow 0,$$

where the definition of ϑ only refers to the cycles of the Koszul complex.

Of course, if $d_q(G_q) \subseteq \mathfrak{m}^2 G_{q-1}$, then $\mathrm{gr}_F(d_q) = 0$, and so all information about G may be lost when passing to $\mathrm{gr}_F(G)$. Much more information is preserved in the spectral sequence $E(G)$ which is derived from the filtered complex G.

As a main result (Theorem 1.1) of this paper we prove that $E(G)$ is isomorphic, up to a shift by one, to the spectral sequence $E(L)$ associated with the filtered complex $L = L(M)$, where $L_q = M \otimes_R K_q \otimes_R S$, $K_q = \Lambda^q(F)$ is the qth exterior power of F and $S = \mathrm{Sym}(F)$ is the symmetric algebra of F, and where F is a free R module with basis $\boldsymbol{e} = e_1, \ldots, e_n$. Note that the sequence $\boldsymbol{x}$ and $\boldsymbol{e}$ operate naturally on $M \otimes S$. So one has two Koszul differentials ∂' and ∂'', and the differential ∂ on L is defined to be $\partial = \partial' + \partial''$. The filtration F on L is given by $F_p L_q = M \otimes K_q \otimes (\bigoplus_{t \geq -p} S_t)$. Here, of course, S_t is the tth symmetric power of F. In particular, $S_t = 0$ for $t < 0$.

It turns out that $E^0(L)$ is isomorphic to the Koszul complex $K(\boldsymbol{x}; M \otimes S)$, and the differentials $d^r : E^r(L) \to E^r(L)$ can again be described in terms of cycles of $K(\boldsymbol{x}; M)$. Thus due to the isomorphism of spectral sequences the Koszul cycles of M determine $E(G)$ as well. We use this isomorphism to prove (Theorem 2.2) that the Koszul cycles of M determine the 'pure part' of G which is defined as follows: For each $q > 0$ let r_q be the largest integer such that $d(G_q) \subseteq \mathfrak{m}^{r_q+1} G_{q-1}$. Then d induces a homogeneous map $\mathrm{gr}_{\mathfrak{m}}(G_q)(-r_q - 1) \to \mathrm{gr}_{\mathfrak{m}}(G_{q-1})$, and one obtains the graded complex $P(G)$

$$\cdots \longrightarrow \mathrm{gr}_{\mathfrak{m}}(G_2)(-a_2) \longrightarrow \mathrm{gr}_{\mathfrak{m}}(G_1)(-a_1) \longrightarrow \mathrm{gr}_{\mathfrak{m}}(G_0) \longrightarrow 0$$

of free $\mathrm{gr}_{\mathfrak{m}}(R)$-modules, where $a_q = \sum_{t=1}^{q} r_t + q$ for $q > 0$.

We say that M has a pure resolution if the preceeding complex is acyclic, in which case it is a minimal graded free resolution of $\mathrm{gr}_{\mathfrak{m}}(M)$.

The whole theory can be formulated correspondingly for graded modules defined over the polynomial ring $R = k[X_1, \ldots, X_n]$. In this situation, it is common to say that a graded module has a pure resolution G of type $(a_1, a_2, \ldots)$ if G is of the form

$$\cdots \longrightarrow R^{b_2}(-a_2) \longrightarrow R^{b_1}(-a_1) \longrightarrow R^{b_0} \longrightarrow 0.$$

It is clear that $P(G) = G$ if M has a pure resolution in the common sense. But the converse need not to be the case. For instance, if we let $R = k[X_1, X_2]$ and M the ideal (X_1^2, X_2^3), then

$$P(G): 0 \longrightarrow R(-2) \xrightarrow{(X_1^2,\, 0)} R^2 \longrightarrow 0$$

is acyclic but $P(G) \neq G$. So our notion of pure resolution is somewhat more general.

For a Stanley–Reisner ring $k[\Delta]$ of a simplicial complex Hochster [8] gave an interpretation of the Koszul homology $H(\boldsymbol{x}; k[\Delta])$ in terms of reduced simplicial cohomology. Thus the resolution of a Stanley–Reisner ring, provided it is pure, is given 'explicitly' as

$$\cdots \xrightarrow{\partial} \bigoplus_W \tilde{H}^{|W|-2}(\Delta_W; k) \otimes_k R \xrightarrow{\partial} \bigoplus_W \tilde{H}^{|W|-1}(\Delta_W; k) \otimes_k R \longrightarrow 0.$$

Here Δ_W is the restriction of Δ to the subset W of the vertex set, $\tilde{H}$ denotes reduced simplicial cohomology and ∂ is defined in terms of simplicial cocycles; see Section 3. It is an open question whether such a resolution can be constructed for general simplicial complexes as well. Note that explicit minimal free resolutions of rings defined by monomials are still missing – see however [11,4].

Examples of Stanley–Reisner rings with pure resolution to which our theory applies are given in a recent paper of Bruns and Hibi [2]. A simple class of such examples are 'forests' which are the one-dimensional simplicial complexes having no cycles. Its simple components are called the 'trees'. We conclude our paper with the 'resolution of forests'.

1. An isomorphism of spectral sequences

In this section we will prove the announced isomorphism of the spectral sequences $E(G)$ and $E(L)$.

For the sake of completeness let us first give the explicit formulas for $\partial' : M \otimes K_q \otimes S_p \to M \otimes K_{q-1} \otimes S_p$ and $\partial'' : M \otimes K_q \otimes S_p \to M \otimes K_{q-1} \otimes S_{p+1}$ whose definition we only indicated in the introduction:

$$\partial'\left(\sum_{1 \le i_1 < \cdots < i_q \le n} m_{i_1,\ldots,i_q} e_{i_1} \wedge \cdots \wedge e_{i_q} \otimes a\right)$$

$$= \sum_j \sum_{1 \le i_1 < \cdots < i_q \le n} (-1)^{j+1} x_{i_j} m_{i_1,\ldots,i_q} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_q} \otimes a$$

and

$$\partial''\left(\sum_{1 \le i_1 < \cdots < i_q \le n} m_{i_1,\ldots,i_q} e_{i_1} \wedge \cdots \wedge e_{i_q} \otimes a\right)$$

$$= \sum_j \sum_{1 \le i_1 < \cdots < i_q \le n} (-1)^{j} m_{i_1,\ldots,i_q} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_q} \otimes e_{i_j} a.$$

Here the 'coefficients' $m_{i_1,\ldots,i_q}$ belong to M, and a belongs to S_p. One easily checks that $\partial'\partial'' + \partial''\partial' = 0$, so that indeed $\partial\partial = 0$. (Note that in the definition of ∂'' we have chosen the sign $(-1)^j$ which for later calculations is more convenient than the other possible choice $(-1)^{j+1}$.)

Both spectral sequences, $E(G)$ and $E(L)$, arise from filtrations. Let us briefly recall the definition of the spectral sequence associated with a filtered complex. So let $(C, d): \cdots \to C_q \to C_{q-1} \to \cdots$ be a filtered complex with filtration F, i.e. a family of submodules $\cdots \subseteq F_{p-1}C_q \subseteq F_pC_q \subseteq \cdots$ for each q such that $d(F_pC_q) \subseteq F_pC_{q-1}$ for all p and q.

For all $r \geq 0$ we set

$$Z^r_{p,q} = \{a \in F_pC_{p+q}: da \in F_{p-r}C_{p+q-1}\}$$

and

$$E^r_{p,q}(C) = (Z^r_{p,q} + F_{p-1}C_{p+q})/(dZ^{r-1}_{p+r-1,q-r+2} + F_{p-1}C_{p+q}).$$

The differential $d^r_{p,q}: E^r_{p,q}(C) \to E^r_{p-r,q+r-1}(C)$ is induced by d: An element of $E^r_{p,q}(C)$ is the homology class $[a]$ of an element $a \in Z^r_{p,q}$. Since $da \in F_{p-r}C_{p+q-1}$, and since $d(da) = 0$, one has $da \in Z^r_{p-r,q+r-1}$, and so one sets $d^r_{p,q}([a]) = [da]$ which is the homology class of da in $E^r_{p-r,q+r-1}(C)$.

The result of these constructions is the spectral sequence $(E^r(C), d^r)_{r \geq 0}$, and one has that $(E^r(C), d^r)$ is a complex with $E^{r+1}(C) = H(E^r(C))$ for all $r \geq 0$.

The set

$$Z^i(E^r_{p,q}(C)) = \{[a] \in E^r_{p,q}(C): da \in F_{p-r-i}C_{p+q-1}\}$$

is the module of elements of $E^r_{p,q}(C)$ which vanish of order i. Given for a fixed r the modules $Z^i(E^r_{p,q}(C))$ and the homomorphisms $d^{i,r}: Z^i(E^r_{p,q}(C)) \to E^r_{p-r-i,q+r+i-1}(C)$, $d^{i,r}[a] \mapsto [da]$, one recovers the spectral sequence from r on. We will use this fact in the proof of the next theorem.

Let us return to the situation described in the introduction. Since R is regular, we have $\mathrm{gr}_{\mathfrak{m}}(R) \cong k \otimes_R S$, and since $G_i \otimes k \cong \mathrm{Tor}_i^R(M,k) \cong H_i(\boldsymbol{x}; M)$ for all i, we obtain natural isomorphisms

$$\begin{aligned} E^1_{p,q}(L) &\cong H_{p+q}(\boldsymbol{x}; M) \otimes_R S_{-p} \cong (G_{p+q} \otimes_R k) \otimes_k \mathfrak{m}^{-p}/\mathfrak{m}^{-p+1} \\ &\cong \mathfrak{m}^{-p}G_{p+q}/\mathfrak{m}^{-p+1}G_{p+q} \cong E^0_{-q,p+2q}(G) \end{aligned}$$

for all p, q. We call these isomorphisms $\varphi^1_{p,q}$, or simply φ.

This can be generalized to the higher E-terms:

Theorem 1.1. *For all $r \geq 1$ there exist isomorphisms*

$$\varphi^r_{p,q}: E^r_{p,q}(L) \longrightarrow E^{r-1}_{-q,p+2q}(G)$$

such that the following diagrams:

$$\begin{array}{ccc} E^r_{p,q}(L) & \xrightarrow{\quad d^r \quad} & E^r_{p-r,q+r-1}(L) \\ {\scriptstyle\varphi^r}\downarrow & & \downarrow{\scriptstyle\varphi^r} \\ E^{r-1}_{-q,p+2q}(G) & \xrightarrow{\quad d^{r-1} \quad} & E^{r-1}_{-q-r+1,p+2q+r-2}(G). \end{array}$$

commute. In other words, the spectral sequences $E^r(L)$ and $E^r(G)$ are isomorphic up to a shift of r by 1.

Proof. For all p,q we have to show that the isomorphism $\varphi^1_{p,q}: E^1_{p,q}(L) \to E^0_{-q,p+2q}(G)$ induces isomorphisms

$$\varphi^1_{p,q}|_{Z^i}: Z^i(E^1_{p,q}(L)) \longrightarrow Z^i(E^0_{-q,p+2q}(G)), \qquad i = 1,2, \ldots$$

such that the diagrams

$$\begin{tikzcd} Z^i(E^1_{p,q}(L)) \arrow[r, "d^{i,1}"] \arrow[d, "\varphi^1_{p,q}|_{Z^i}"'] & E^1_{p-i-1,q+i}(L) \arrow[d, "\varphi^1_{p,q}"] \\ Z^i(E^0_{-q,p+2q}(G)) \arrow[r, "d^{i,0}"] & E^0_{-q-i,p+2q+i-1}(G) \end{tikzcd}$$

commute.

Simplifying notation we have to show that for all j and k the following assertion (*) holds: The map

$$\varphi: H_j(\boldsymbol{x}; M) \otimes S_k \to \mathfrak{m}^k G_j/\mathfrak{m}^{k+1} G_j$$

induces an isomorphism between the submodule

$$Z^i_{jk}(M) \subseteq H_j(\boldsymbol{x}; M) \otimes S_k$$

of elements of $H_j(\boldsymbol{x}; M) \otimes S_k$ vanishing of order i, and the submodule

$$U^i_{jk}(M) \subseteq \mathfrak{m}^k G_j/\mathfrak{m}^{k+1} G_j$$

of elements of $\mathfrak{m}^k G_j/\mathfrak{m}^{k+1} G_j$ vanishing of order i such that

$$\begin{tikzcd} Z^i_{jk}(M) \arrow[r, "d^{i,1}"] \arrow[d, "\varphi|_{Z^i}"'] & H_{j-1}(\boldsymbol{x}; M) \otimes S_{k+i+1} \arrow[d, "\varphi"] \\ U^i_{jk}(M) \arrow[r, "d^{i,0}"] & \mathfrak{m}^{k+i+1} G_{j-1}/\mathfrak{m}^{k+i+2} G_{j-1} \end{tikzcd} \tag{1}$$

commutes.

The assertion is trivial if $j = 0$, because then $Z^i_{0k}(M) = H_0(\boldsymbol{x}; M) \otimes S_k$, and $U^i_{0k} = \mathfrak{m}^k G_0/\mathfrak{m}^{k+1} G_0$ for all i. In the next steps we give the reduction to $j = 1$. Suppose $j > 1$; let $\Omega^1(M)$ be the first syzygy module of M, and let $\delta: H_j(\boldsymbol{x}; M) \to H_{j-1}(\boldsymbol{x}; \Omega^1(M))$ be the connecting isomorphism arising from the short exact sequence $0 \to \Omega^1(M) \to G_0 \to M \to 0$. Reduction to $j = 1$ is achieved once we have shown:

(a) The diagram

$$\begin{tikzcd} H_j(\boldsymbol{x}; M) \otimes S_k \arrow[r, "\varphi"] \arrow[d, "\delta"'] & \mathfrak{m}^k G_j/\mathfrak{m}^{k+1} G_j \arrow[d, equal] \\ H_{j-1}(\boldsymbol{x}; \Omega^1(M)) \otimes S_k \arrow[r, "\varphi"] & \mathfrak{m}^k G_j/\mathfrak{m}^{k+1} G_j \end{tikzcd} \tag{2}$$

is commutative up to a sign. More precisely, $\varphi\delta = (-1)^j\varphi$. Furthermore, $\delta(Z^i_{jk}(M)) = Z^i_{j-1k}(\Omega^1(M))$ and $U^i_{jk}(M) = U^i_{j-1k}(\Omega^1(M))$. (The last equality is trivially true.)

(b) The diagram

$$\begin{array}{ccc} Z^i_{jk}(M) & \xrightarrow{d^{i,1}} & H_{j-1}(x;M)\otimes S_{k+i+1} \\ \delta\downarrow & & \delta\downarrow \\ Z^i_{j-1k}(\Omega^1(M)) & \xrightarrow{d^{i,1}} & H_{j-2}(x;\Omega^1(M))\otimes S_{k+i+1} \end{array} \tag{3}$$

is anti-commutative.

(The similar statement for $U^i_{jk}(M) \xrightarrow{d^{i,0}} \mathfrak{m}^{k+i+1}G_{j-1}/\mathfrak{m}^{k+i+2}G_{j-1}$ is clear.) Let us first verify that (a) and (b) imply the commutativity of (1): To do this we use a more precise notation and set $\varphi_{M,j}$ for the homomorphism $\varphi: H_j(x;M)\otimes S_k \to \mathfrak{m}^kG_j/\mathfrak{m}^{k+1}G_j$.Then, by (a), $(-1)^j\varphi_{M,j} = \varphi_{\Omega^1(M),j-1}\delta$, so that if we assume the commutativity of (1) for $j-1$, it follows together with (b) that

$$\begin{aligned} d^{i,0}\varphi_{M,j} &= (-1)^j d^{i,0}\varphi_{\Omega^1(M),j-1}\delta = (-1)^j\varphi_{\Omega^1(M),j-2}d^{i,1}\delta \\ &= (-1)^j\varphi_{\Omega^1(M),j-2}(-\delta d^{i,1}) = (-1)^{j-1}\varphi_{\Omega^1(M),j-1}\delta d^{i,1} = \varphi_{M,j-1}d^{i,1}. \end{aligned}$$

Proof of (a): Identifying $\mathfrak{m}^kG_j/\mathfrak{m}^{k+1}G_j$ with $(G_j\otimes_R k)\otimes_R S_k$, we have $\varphi = \alpha\otimes S_k$, where $\alpha: H_j(x;M)\to G_j\otimes_R k$ is the canonical isomorphism, described as follows: Set $K(x) = K(x;R)$ and consider the complex $G\otimes K(x)$. Recall that the differential on $G\otimes K(x)$ is the tensor product $d\otimes t$ of the differentials on G and $K(x)$, which for an element $a\otimes b\in G_i\otimes K_j(x)$ is defined by the equation $(d\otimes t)(a\otimes b) = da\otimes b + (-1)^i a\otimes tb$.

Let $z\in K_j(x;M)$ be a cycle representing a homology class in $H_j(x;M)$. Since the rows $G\otimes K_i(x)$ of $G\otimes K(x)$ are acyclic one finds a cycle $\tilde{z}\in(G\otimes K(x))_j = \bigoplus_{i=0}^j G_i\otimes K_{j-i}(x)$ with $\tilde{z} = z_0 + \cdots + z_j$, $z_i\in G_i\otimes K_{j-i}(x)$ and $z = (\varepsilon\otimes K_j(x))(z_0)$, where $\varepsilon: G_0\to M$ is the canonical epimorphism.

Note that $z_j\in G_j\otimes K_0(x) = G_j$, and $\alpha([z]) = \bar{z}_j$, where overbar denotes the canonical epimorphism $G_j\to G_j\otimes k$.

Now let us prove that diagram (2) commutes up to a sign. It suffices to show that

$$\begin{array}{ccc} H_j(x;M) & \xrightarrow{\alpha} & G_j\otimes k \\ \delta\downarrow & & \| \\ H_{j-1}(x;\Omega^1(M)) & \xrightarrow{\alpha} & G_j\otimes k \end{array}$$

has this property.

With the notation just introduced we have $\delta([z]) = [(G\otimes t)(z_0)]$. Observe that

$$\tilde{G}: \cdots \longrightarrow G_2 \longrightarrow G_1 \longrightarrow 0$$

is the resolution of $\Omega^1(M)$. In order to apply α to $\delta[z] = [(G\otimes t)(z_0)]$ we have to lift $(G\otimes t)(z_0)$ to a cycle in $\tilde{G}\otimes K(x)$. Obviously, $-z_1 + z_2 - \cdots +(-1)^j z_j$ is such a lifting, and so

$$\alpha\delta[z] = (-1)^j \bar{z}_j = ((-1)^j \alpha[z]$$

as desired.

We now show that $\delta(Z^i_{jk}(M)) = Z^i_{j-1k}(\Omega^1(M))$. For this we must first note that $[a] \in H_j(\boldsymbol{x}; M) \otimes S_k$ belongs to $Z^i_{jk}(M)$ if there exists $\tilde{a} = \sum_{t \geq k} a_t \in F_{-k}L_j = \bigoplus_{t \geq k} K_j(\boldsymbol{x}; M) \otimes S_t$, $a_t \in K_j(\boldsymbol{x}; M) \otimes S_t$, such that $a_k = a$ and $\partial(\tilde{a}) \in F_{-k-i-1}L_{j-1}$. Equivalently, $[a] = [\tilde{a}]$ (in $E^1(L)$) and $\partial'' a_{k+s-1} + \partial' a_{k+s} = 0$ for $s = 1, \dots, i$.

From now on we will denote, without any danger of confusion, all maps induced by the augmentation map $\varepsilon: G_0 \to M$ again by ε. For example, in the following diagram the map $F_{-k}L_j(G_0) \to F_{-k}L_j(M)$ will be called ε.

The exact sequence $0 \to \Omega^1(M) \to G_0 \to M \to 0$ yields the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & F_{-k}L_j(\Omega^1(M)) \arrow[r] \arrow[d, "\partial"] & F_{-k}L_j(G_0) \arrow[r] \arrow[d, "\partial"] & F_{-k}L_j(M) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & F_{-k}L_{j-1}(\Omega^1(M)) \arrow[r] & F_{-k}L_{j-1}(G_0) \arrow[r] & F_{-k}L_{j-1}(M) \arrow[r] & 0
\end{tikzcd}$$

of complexes whose rows are exact.

Thus there exists $\tilde{b} = \sum_{t \geq k} b_t \in F_{-k}L_j(G_0)$, $b_t \in K_j(\boldsymbol{x}; G_0) \otimes S_t$, with $\varepsilon\tilde{b} = \tilde{a}$. The element $\partial\tilde{b}$ does not necessarily belong to $F_{-k}L_{j-1}(\Omega^1(M))$. But since $\varepsilon\partial\tilde{b} \in F_{-k-i-1}L_{j-1}(M)$, we see that the element $\tilde{c} = \partial' b_k + (\partial'' b_k + \partial' b_{k+1}) + \cdots + (\partial'' b_{k+i-1} + \partial' b_{k+i})$ belongs to $F_{-k}L_{j-1}(\Omega^1(M))$. Furthermore, as $\delta[a] = \delta([a_k]) = [\partial' b_k]$, and as $\partial\tilde{c} = \partial''\partial' b_{k+i} \in F_{-k-i-1}L_{j-2}(\Omega^1))$ we have $\delta[a] \in Z^i_{j-1k}(\Omega^1(M))$.

Conversely, given $[c] \in Z^i_{j-1k}(\Omega^1(M))$ we want to find $[a] \in Z^i_{jk}(M)$ such that $\delta[a] = [c]$. There exists $\tilde{c} = \sum_{t \geq k} c_t \in F_{-k}L_{j-1}(\Omega^1(M))$, $c_t \in K_{j-1}(\boldsymbol{x}; \Omega^1(M)) \otimes S_t$, such that $c_k = c$ and $\partial(\tilde{c}) \in F_{-k-i-1}L_{j-2}(\Omega^1(M))$. We construct $\tilde{b} = \sum_{t=k}^{k+i} b_t$ with $b_t \in K_j(\boldsymbol{x}; G_0) \otimes S_t$ such that $\partial\tilde{b} = \sum_{t=k}^{k+i} c_t + \partial'' b_{k+i}$. As c_k is a cycle in $K_{j-1}(\boldsymbol{x}; G_0) \otimes S_k$ and $K(x; G_0) \otimes S_k$ is acyclic, there exists $b_k \in K_j(\boldsymbol{x}; G_0) \otimes S_k$ with $\partial' b_k = c_k$. If $i = 0$, then $\tilde{b} = b_k$. If $i > 0$, then $\partial'' c_k + \partial' c_{k+1} = 0$. So $\partial' c_{k+1} = -\partial''\partial'(b_k) = \partial'\partial'' b_k$, and hence $\partial'(c_{k+1} - \partial'' b_k) = 0$. Since $K(\boldsymbol{x}; G_0) \otimes S_{k+1}$ is acyclic we find $b_{k+1} \in K_j(\boldsymbol{x}; G_0) \otimes S_{k+1}$ with $\partial' b_{k+1} = c_{k+1} - \partial'' b_k$, so that $c_{k+1} = \partial'' b_k + \partial' b_{k+1}$. Thus induction on i yields the construction of $\tilde{b}$. Set $a_t = \varepsilon b_t$ for $t = k, \dots, k+i$, and $\tilde{a} = \sum_{t=k}^{k+i} a_t$. By construction, $\partial\tilde{a} = \varepsilon\partial'' b_{k+i} \in F_{-k-i-1}L_{j-1}(M)$, so that $[a_k] \in Z^i_{jk}(M)$, and $\delta[a_k] = [c]$, as desired.

Proof of (b): Given $[a] \in Z^i_{jk}(M)$, we choose $\tilde{a} = \sum_{t \geq k} a_t$ and $\tilde{b} = \sum_{t \geq k} b_t$ as in the proof of (a). Then $\delta[a] = [\partial' b_k]$, and $d^{i,1}[a] = [\partial'' a_{k+i}]$. Further $d^{i,1}[\partial' b_k] = [\partial''(\partial'' b_{k+i-1} + \partial' b_{k+i})] = [\partial''\partial' b_{k+i}]$, so that

$$\begin{aligned}
\delta d^{i,1}[a] &= \delta[\partial'' a_{k+i}] = [\partial'\partial'' b_{k+i}] = -[\partial''\partial' b_{k+i}] \\
&= -d^{i,0}[\partial' b_k] = -d^{i,0}\delta[a],
\end{aligned}$$

as asserted.

In order to complete the proof of the theorem it remains to prove assertion (*) for $j = 1$.

Consider the isomorphism $\beta: G_j \otimes k \to H_j(\boldsymbol{x}; M)$ which is inverse to the map α from part (a) of the proof. The map β induces the isomorphism $\psi = \beta \otimes S_k: (G_j \otimes_R k) \otimes_R S_k = G_j \otimes_R \mathfrak{m}^k/\mathfrak{m}^{k+1} \to H_j(\boldsymbol{x}; M) \otimes_R S_k$. Note that ψ is inverse to φ.

To simplify notation we set $U_k^i = U_{1k}^i(M)$ and $Z_k^i = Z_{1k}^i(M)$, and prove:

(c) $\psi(U_k^i) \subseteq Z_k^i$, and diagram (1) is commutative (for $j = 1$),

(d) $\varphi(Z_k^i) \subseteq U_k^i$.

Proof of (c): We need to describe β explicitly: Let $G_1 = \bigoplus_j Rg_i$. Since $dg_j \in \mathfrak{m}G_0$, we have $dg_j = -\sum_{q=1}^n x_q h_{qj}$, $h_{qj} \in G_0$. Set $z_j = \sum_{q=1}^n h_{qj} e_q + g_j \in (G \otimes K(\boldsymbol{x}))_1 = K_1(\boldsymbol{x}; G_0) \oplus G_1$. Then z_j is a cycle in $G \otimes K(\boldsymbol{x})$, and so if we set $b_j = \sum_s \varepsilon(h_{sj}) e_s$, then $\partial' b_j = 0$, and

$$\beta(g_j \otimes \bar{1}) = [b_j] \in H_1(\boldsymbol{x}; M).$$

Here $\varepsilon: G_0 \to M$ is, as before, the augmentation map.

Next let us describe ψ: Let $s \in \mathfrak{m}^k G_1$. Then $s = \sum_j f_j g_j$ with $f_j \in \mathfrak{m}^k$. One can write $f_j = \sum_I r_{Ij} x^I$, with $x^I = x_1^{i_1} \cdots x_n^{i_n}$, $i_1 + \cdots + i_n = k$, $r_{Ij} \in R$. Set $f_j(e) = \sum r_{Ij} e^I$, $e^I = e_1^{i_1} \cdots e_n^{i_n}$. Then $[s] = s \bmod \mathfrak{m}^{k+1} G_1$ can be identified with $\sum_j g_j \otimes \bar{1} \otimes f_j(e)$. Hence

$$\psi[s] = \sum_j \beta(g_j \otimes \bar{1}) \otimes f_j(e) = \sum_j [b_j] \otimes f_j(e).$$

Thus if we set $a_k = \sum_j b_j \otimes f_j(e)$, then $\psi[s] = [a_k]$.

Consider

$$\eta = \eta_t: G_0 \otimes S_t = K_0(\boldsymbol{x}; G_0) \otimes S_t \to \mathfrak{m}^t G_0, \quad \eta(h \otimes e^I) = x^I h.$$

Then we obtain an anti-commutative diagram

$$\begin{array}{ccc} K_1(\boldsymbol{x}; G_0) \otimes S_t & \xrightarrow{\partial''} & G_0 \otimes S_{t+1} \\ \downarrow{\scriptstyle \partial'} & & \downarrow{\scriptstyle \eta_{t+1}} \\ \mathfrak{m}G_0 \otimes S_t & \xrightarrow{\eta_t} & \mathfrak{m}^{t+1} G_0 \end{array} \tag{4}$$

Indeed, for $c = he_q \otimes e^I$ one has $\eta_t \partial' c = \eta_t(x_q h \otimes e^I) = x^I x_q h$, and $\eta_{t+1} \partial'' c = \eta_{t+1}(-h \otimes e_q e^I) = -x^I x_q h$.

Returning to the description of β and ψ we set $l_j = \sum_{q=1}^n h_{qj} e_q$ and $h_k = \sum_j l_j \otimes f_j(e)$. Then $\partial' l_j = -dg_j$, $a_k = \varepsilon h_k$ and

$$\begin{aligned} ds = \sum_j f_j dg_j = -\sum_j f_j \partial' l_j = -\eta_k \left(\sum_j \partial' l_j \otimes f_j(e) \right) \\ = -\eta_k \partial' h_k = \eta_{k+1} \partial'' h_k. \end{aligned}$$

The following statement, which will be shown by induction on i, will imply that $\psi(U_k^i) \subseteq Z_k^i$: Let $s \in \mathfrak{m}^k G_1$ be such that $ds \in \mathfrak{m}^{k+i+1} G_0$. Then there exists an element

$a = \sum_{t=k}^{k+i} a_t$, where $a_t \in K_1(\boldsymbol{x}; M) \otimes S_t$ and such that $\partial'' a_{t-1} + \partial' a_t = 0$ for $t = k, \dots, k+i$ and $\psi[s] = [a]$. Moreover, if $a_t = \varepsilon h_t$ with $h_t \in K_1(\boldsymbol{x}; G_0) \otimes S_t$, $t = k, \dots, k+i$, then $ds = \eta \partial'' h_{k+i}$.

The above calculations prove the assertion for $i = 0$. So assume that it is true for i, and let $s \in \mathfrak{m}^k G_1$ be such that $ds \in \mathfrak{m}^{k+i+2} G_0$. Since $ds \in \mathfrak{m}^{k+i+2} G_0$, by induction there exists an element $a = \sum_{t=k}^{k+i} a_t$ with the above properties. Then $\eta \partial'' h_{k+i} = ds \in \mathfrak{m}^{k+i+2} G_0$, so that $\partial'' h_{k+i} \in \mathfrak{m} G_0 \otimes S_{k+i+1}$. Therefore, $\partial'' a_{k+i} = \partial'' \varepsilon h_{k+i} = \varepsilon \partial'' h_{k+i} \in \mathfrak{m} M \otimes S_{k+i+1} = Im\, \partial'$. Hence there exists an element $a_{k+i+1} \in K_1(\boldsymbol{x}; M) \otimes S_{k+i+1}$ such that $\partial'' a_{k+i} = -\partial' a_{k+i+1}$. Choose h_{k+i+1} with $a_{k+i+1} = \varepsilon h_{k+i+1}$. Then $\partial'' a_{k+i} + \partial' a_{k+i+1} = 0$ implies $\partial'' h_{k+i} + \partial' h_{k+i+1} \in \operatorname{Ker} \varepsilon = \operatorname{Im} d \otimes S_{k+i+1}$, so that $\partial'' h_{i+k} + \partial' h_{k+i+1} = \sum_j dg_j \otimes p_{jk+i+1}$ with $p_{jk+i+1} \in S_{k+i+1}$. Set $y_{k+i+1} = \sum_j l_j \otimes p_{jk+i+1}$. Then $\sum_j dg_j \otimes p_{jk+i+1} = -\partial' y_{k+i+1}, \varepsilon \partial' y_{k+i+1} = 0$, and for $\tilde{h}_{k+i+1} = h_{k+i+1} + y_{k+i+1}$ one has: $\partial'' h_{k+i} = -\partial' \tilde{h}_{k+i+1}, ds = \eta \partial'' h_{k+i} = -\eta \partial' \tilde{h}_{k+i+1} = \eta \partial'' \tilde{h}_{k+i+1}$.

Set $\tilde{a}_{k+i+1} = \varepsilon \tilde{h}_{k+i+1}$, and consider $\tilde{a} = a_k + \cdots + a_{k+i} + \tilde{a}_{k+i+1}$. Then $\partial' \tilde{a}_{k+i+1} = \partial' a_{k+i+1} + \partial' \varepsilon y_{k+i+1}$. But $\partial' \varepsilon y_{k+i+1} = \varepsilon \partial' y_{k+i+1} = 0$, so that $\partial' \tilde{a}_{k+i+1} = \partial' a_{k+i+1} = -\partial'' a_{k+i}$. Thus $[\tilde{a}] \in Z_k^{i+1}$. Since $[\tilde{a}] = [a] = \psi[s]$, the element $\tilde{a}$ has the required properties, and $ds = \eta \partial'' \tilde{h}_{k+i+1}$.

As a corollary of these computations we also obtain the following commutative diagram:

$$\begin{array}{ccc} U_k^i & \xrightarrow{d^{i,0}} & \mathfrak{m}^{k+i+1} G_0 / \mathfrak{m}^{k+i+2} G_0 \\ \psi \downarrow & & \downarrow \psi \\ Z_k^i & \xrightarrow{d^{i,1}} & H_0(\boldsymbol{x}; M) \otimes S_{k+i+1}. \end{array}$$

Indeed, let $[s] \in U_k^i$. Then $\psi[s] = [a]$ where $a = \sum_{t=k}^{k+i} a_t$, and $\partial a = \partial'' a_{k+i} = \partial'' \varepsilon h_{k+i}$. Moreover, since $ds = \eta \partial'' h_{k+i}$, we get

$$\psi d^{i,0}[s] = \psi[ds] = \psi[\partial'' h_{k+i}] = [\varepsilon \partial'' h_{k+i}] = [\partial'' \varepsilon h_{k+i}] = d^{i,1}[a].$$

The second equality is valid since $[\partial'' h_{k+i}] = [\eta \partial'' h_{k+i}]$ after identification of $(G_0 \otimes S_{k+i+1})/(\mathfrak{m} G_0 \otimes S_{k+i+1})$ with $\mathfrak{m}^{k+i+1} G_0 / \mathfrak{m}^{k+i+2} G_0$ via η.

Since ψ is inverse to φ, this commutative digram together with statement (d) yields the commutativity of (1) and thus completes the proof of (c).

Proof of (d): We prove by induction on i the following statement. Let $a = \sum_{t \geq k} a_t$, $a_t \in K_1(\boldsymbol{x}; M) \otimes S_t$, be such that $[a] \in Z_k^i$. Then $\varphi[a] \in U_k^i$. Moreover, there exists an element $s \in \mathfrak{m}^k G_1$ such that $\varphi[a] = [s]$ and $ds = \eta \partial'' h_{k+i}$ where $\varepsilon h_{k+i} = a_{k+i}$, $h_{k+i} \in K_1(\boldsymbol{x}; G_0) \otimes S_{k+i}$.

If $i = 0$ we need only to show that there exists $s \in \mathfrak{m}^k G_1$ with $\varphi[a_k] = [s]$ and $ds = \eta \partial'' h_k$. So let $a_k = \sum_I b_I \otimes e^I \in K_1(\boldsymbol{x}; M) \otimes S_k$. Write $b_I = \sum_{q=1}^n m_{qI} e_q$, $m_{qI} \in M$, and choose $h_{qI} \in G_0$ with $m_{qI} = \varepsilon h_{qI}$. Then $\sum_{q=1}^n x_q h_{qI} \in \operatorname{Ker} \varepsilon = \operatorname{Im} d$, and hence $\sum_{q=1}^n x_q h_{qI} = -dg_I$ for certain $g_I \in G_1$. Set $s = \sum_I x^I g_I$, then $\varphi[a_k] = [s]$. Now if we set $h_k = \sum_I \sum_q h_{qI} e_q \otimes e^I$, then $\varepsilon h_k = a_k$, and $ds = \sum_I x^I dg_I = -\sum_I \sum_q x^I x_q h_{qI} = \eta_{k+1} \partial'' h_k$, as desired.

Now assume the assertion is true for i, and take $a = \sum_{t \geq k} a_t$ such that $[a] \in Z_k^{i+1}$. Since $[a] \in Z_k^i$, there exists an element s as above such that $\varphi[a] = [s]$. Choose $h_{k+i+1} \in K_1(\boldsymbol{x}; G_0) \otimes S_{k+i+1}$ such that $a_{k+i+1} = \varepsilon h_{k+i+1}$. Since $\partial'' a_{k+i} + \partial' a_{k+i+1} = 0$, one has $\partial'' h_{k+i} + \partial' h_{k+i+1} \in \operatorname{Ker} \varepsilon = \operatorname{Im} d \otimes S_{k+i+1}$. Therefore, $\partial'' h_{k+i} + \partial' h_{k+i+1} = \sum_j dg_j \otimes p_{jk+i+1}$ for some $p_{jk+i+1} \in S_{k+i+1}$. Applying η, one obtains $\eta \partial'' h_{k+i} - \sum p_{jk+i+1}(x) dg_j = -\eta \partial' h_{k+i+1}$, and since by assumption $ds = \eta \partial'' h_{k+i}$, one has $d(s - \sum p_{jk+i+1}(x) g_j) = -\eta \partial' h_{k+i+1} = \eta \partial'' h_{k+i+1}$.

Set $\tilde{s} = s - \sum p_{jk+i+1}(x) g_j$. Then $[\tilde{s}] = [s]$ and $d\tilde{s} = \eta \partial'' h_{k+i} \in \mathfrak{m}^{k+i+2} G_0$. This completes the proof of the theorem. $\square$

2. The pure part of a resolution

Let $(E^r_{p,q}, d^r_{p,q})_{r \geq 0}$ be an arbitrary spectral sequence (notation as in [9, Chapter XI.1]). For all $t \in \mathbb{Z}$ we set

$$E^r_t = \bigoplus_{p+q=t} E^r_{p,q}, \qquad d^r_t = \bigoplus_{p+q=t} d^r_{p,q}.$$

Then

$$\cdots \longrightarrow E^r_q \xrightarrow{d^r_t} E^r_{t-1} \xrightarrow{d^r_{t-1}} E^r_{t-2} \longrightarrow \cdots$$

is a complex for all $r \geq 0$.

For each t, let r_t be the least integer such that $d^{r_t}_t \neq 0$. We set

$$Z^i(E^r_t) = \bigoplus_{p+q=t} Z^i(E^r_{p,q}), \qquad B^i(E^r_t) = \bigoplus_{p+q=t} B^i(E^r_{p,q}).$$

Here $Z^i(E^r_{p,q}) = Z^{r+i}_{p,q}/B^r_{p,q}$ and $B^i(E^r_{p,q}) = B^{r+i}_{p,q}/B^r_{p,q}$.

Since $Z^i(E^0_t) = E^0_t$ and $B^i(E^0_{t-1}) = 0$ for all $i \leq r_t$, we may define the homomorphism $E^0_t \to E^0_{t-1}$ as the composition

$$E^0_t \longrightarrow E^0_t / B^{r_t}(E^0_t) = Z^{r_t}(E^0_t)/B^{r_t}(E^0_t) = E^{r_t}_t$$

$$\xrightarrow{d^{r_t}_t} E^{r_t}_{t-1} = Z^{r_t}(E^0_{t-1})/B^{r_t}(E^0_{t-1}) = Z^{r_t}(E^0_{t-1}) \longrightarrow E^0_{t-1}.$$

We denote this homomorphism (which is essentially $d^{r_t}_t$) again by $d^{r_t}_t$.

Observe that $(E^0_t, d^{r_t}_t)_{t \in \mathbb{Z}}$ is a complex. Indeed,

$$\operatorname{Im} d^{r_t}_t = B^{r_t+1}(E^0_{t-1}) \subseteq Z^{r_{t-1}+1}(E^0_{t-1}) = \operatorname{Ker} d^{r_t}_t.$$

Let $P(G)$ be the pure part of G, as described in the introduction. Then

Propositon 2.1. $P(G) \cong (E_t^0(G), d_t^{r_t})_{t\in\mathbb{Z}}$.

Proof. We have

$$E_t^0(G) = \bigoplus_{p+q=t} E_{p,q}^0(G) = \bigoplus_{p+q=t} F_pG_{p+q}/F_{p-1}G_{p+q} = \oplus \mathfrak{m}^{-(p+t)}G_t/\mathfrak{m}^{-(p+t)+1}G_t.$$

So we see that $E_{p,t+p}^0(G) = 0$ for $p > -t$, and that $E_{p,t-p}^0(G) \neq 0$ for $p \leq -t$. In particular, if $p > -t$, then $d_{p,t-p}^r = 0$ for all r.

We claim, that if $d_{-t,2t}^r = 0$ for all $r \leq s$, then $d_{p,t-p}^r = 0$ for all $r \leq s$ and all $p \leq -t$. Indeed, the hypothesis implies that $dG_t \subseteq \mathfrak{m}^{s+2}G_{t-1}$. But then we have $d(\mathfrak{m}^{-(p+t)}G_t) \subseteq \mathfrak{m}^{-(p+t)+s+2}G_{t-1}$ for all $p \leq t$, and this implies $d_{p,t-p}^r = 0$ for all $r \leq s$.

As a consequence of the claim we get

$$r_t = \min\{r \mid d_{-t,2t}^r \neq 0\} = \max\{r \mid d(G_t) \subseteq \mathfrak{m}^{r+1}G_{t-1}\}.$$

Furthermore, $d_t^{r_t}: E_t^0(G) \to E_{t-1}^0(G)$ is given on the homogeneous components as follows:

$$\mathfrak{m}^iG_t/\mathfrak{m}^{i+1}G_t \to \mathfrak{m}^{i+r_t+1}G_{t-1}/\mathfrak{m}^{i+r_t+2}G_{t-1},$$

$$a + \mathfrak{m}^{i+1}G_t \mapsto da + \mathfrak{m}^{i+r_t+2}G_{t-1}.$$

Here $d_t^{r_t}$ coincides with the differential of $P(G)$. $\square$

Combining proposition 2.1 with our main theorem we obtain

Theorem 2.2. *The pure part $P(G)$ of the resolution G is isomorphic to the complex*

$$\cdots \longrightarrow H_t(\boldsymbol{x}; M) \otimes \mathrm{gr}_m(R)(-a_t) \xrightarrow{\vartheta_t} H_{t-1}(\boldsymbol{x}; M) \otimes \mathrm{gr}_m(R)(-a_{t-1}) \xrightarrow{\vartheta_{t-1}}$$
$$\cdots \xrightarrow{\vartheta_2} H_1(\boldsymbol{x}; M) \otimes \mathrm{gr}_m(R)(-a_1) \xrightarrow{\vartheta_1} H_0(\boldsymbol{x}; M) \otimes \mathrm{gr}_m(R) \longrightarrow 0,$$

where $\vartheta_t = d_t^{r_t+1}$ with r_t as in Proposition 2.1, $a_t = \sum_{i=1}^t r_t + t$, and where d^{r_t+1} is 'essentially' the differential of $E_t^{r_t+1}(L)$.

In more explicit terms, ϑ_t can be described as follows: For $a \in K_t(\boldsymbol{x}; M)$ and $1 \leq i \leq n$ there is a unique decomposition

$$a = \sigma_i a - e_i \wedge \pi_i a, \quad \text{with } \sigma_i a, \pi_i a \in K(\boldsymbol{x}_i; M),\ \boldsymbol{x}_i = x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n,$$

and we have $\partial'' a \otimes h = \sum_i \pi_i a \otimes e_i h$.

Now given $[z] \otimes 1 \in H_t(\boldsymbol{x}; M) \otimes \mathrm{gr}_{\mathfrak{m}}(R)$. We identify $\mathrm{gr}_m(R)$ with $k \otimes_R S = k[e_1, \ldots, e_n]$ and consider $w_1 = \sum_i [\pi_i(z)] \otimes e_i$. If $r_t = 0$, then $\vartheta_t([z] \otimes 1) = w_1$. If $r_t > 0$, then $w_1 = 0$, and we choose $b_i \in K_t(\boldsymbol{x}; M)$ with $\partial' b_i = \pi_i(z)$. Then we form $\partial''(\sum_i b_i \otimes e_i) = \sum_{i,j} \pi_j b_i \otimes e_i e_j$, and consider $w_2 = \sum_{i \leq j} [\pi_i b_j + \pi_j b_i] \otimes e_i e_j$. If $r_t = 1$, then $\vartheta_t([z] \otimes 1) = w_2$. Otherwise, we can find $c_{ij} \in K_t(\boldsymbol{x}; M)$ with $\partial' c_{ij} = \pi_i b_j + \pi_j b_i$, and so on.

Let us demonstrate this construction at two simple examples (whose resolutions of course could be computed easily without any theory):

(i) Consider the graded k-algebra

$$R = k[x_1, \dots, x_4] = k[X_1, \dots, X_4]/(X_1X_3, X_1X_4, X_2X_3, X_2X_4).$$

Then $H_1(\boldsymbol{x}; R)$ is generated by the homology classes $f_1 = [x_1e_3]$, $f_2 = [x_1e_4]$, $f_3 = [x_2e_3]$ and $f_4 = [x_2e_4]$, $H_2(\boldsymbol{x}; R)$ by $g_1 = [x_3e_1 \wedge e_2]$, $g_2 = [x_4e_1 \wedge e_2]$, $g_3 = [x_1e_3 \wedge e_4]$, and $g_4 = [x_2e_3 \wedge e_4]$, and $H_3(\boldsymbol{x}; R)$ by $h = [x_1e_2 \wedge e_3 \wedge e_4 - x_2e_1 \wedge e_3 \wedge e_4]$.

The cycles representing the homology classes are homogeneous, i.e. of the form $\sum_I a_I e_I$, where all $a_I \in R$ are homogeneous of the same degree, say s, and $I = \{i_1, \dots, i_t\}$. The degree of $\sum_I a_I e_I$ is then $s + t$.

Now R has a graded free resolution

$$0 \longrightarrow Sh \xrightarrow{d_2} \bigoplus_{i=1}^{4} Sg_i \xrightarrow{d_1} \bigoplus_{i=1}^{4} Sf_i \longrightarrow S \longrightarrow 0,$$

where $S = k[e_1, e_2, e_3, e_4]$ (the X_i are identified with the e_i). We see from the degrees of the cycles that the resolution is pure of type $(2, 1, 1)$. Let us compute d_2 which is linear:

$$\begin{aligned}
&\partial''((x_1e_2 \wedge e_3 \wedge e_4 - x_2e_1 \wedge e_3 \wedge e_4) \otimes 1) \\
&= -x_1e_3 \wedge e_4 \otimes e_2 + x_1e_2 \wedge e_4 \otimes e_3 - x_1e_2 \wedge e_3 \otimes e_4 \\
&\quad + x_2e_1 \wedge e_3 \otimes e_4 - x_2e_1 \wedge e_4 \otimes e_3 + x_2e_3 \wedge e_4 \otimes e_1 \\
&= x_2e_3 \wedge e_4 \otimes e_1 - x_1e_3 \wedge e_4 \otimes e_2 \\
&\quad + (x_1e_2 \wedge e_4 - x_2e_1 \wedge e_4) \otimes e_3 + (-x_1e_2 \wedge e_3 + x_2e_1 \wedge e_3) \otimes e_4.
\end{aligned}$$

Note that $x_1e_2 \wedge e_4 - x_2e_1 \wedge e_4$ is homologous to $x_4e_1 \wedge e_2$ and that $-x_1e_2 \wedge e_3 + x_2e_1 \wedge e_3$ is homologous to $-x_3e_1 \wedge e_2$, so that

$$d_2h = -e_4g_1 + e_3g_2 - e_2g_3 + e_1g_4.$$

Similarly we could compute d_1.

(ii) In the following example which is equally simple we compute a map of degree two. The $S = k[X_1, X_2]$-module $M = (X_1, X_2)/(X_1^2, X_2^2)$ has the following generators for the Koszul homology: $f_1 = [x_1]$ and $f_2 = [x_2]$ for $H_0(\boldsymbol{x}; M)$, $g_1 = [x_1e_1]$, $g_2 = [x_2g_2]$and $g_3 = [x_1e_2 - x_2e_1]$ for $H_1(\boldsymbol{x}; M)$, $h = [x_1x_2e_1 \wedge e_2]$ for $H_2(\boldsymbol{x}; M)$. Here we have denoted the images of the X_i in M by x_i, $i = 1, 2$. So we see that M has the S-resolution

$$0 \longrightarrow Sh \xrightarrow{d_2} \bigoplus_{i=1}^{3} Sg_i \xrightarrow{d_1} \bigoplus_{i=1}^{2} Sf_i \longrightarrow 0,$$

which is pure of type $(2, 2)$. Let us compute d_2:

$$\partial''(x_1x_2e_1 \wedge e_2) = -x_1x_2e_2 \otimes e_1 + x_1x_2e_1 \otimes e_2.$$

Now $-x_1x_2e_2$ and $x_1x_2e_2$ are boundaries, and so we can write

$$\partial'(-x_2e_1\wedge e_2\otimes e_1 - x_1e_1\wedge e_2\otimes e_2) = -x_1x_2e_2\otimes e_1 + x_1x_2e_1\otimes e_2.$$

Then

$$\partial''(-x_2e_1\wedge e_2\otimes e_1 - x_1e_1\wedge e_2\otimes e_2)$$
$$= x_2e_2\otimes e_1^2 - x_2e_1\otimes e_1e_2 + x_1e_2\otimes e_1e_2 - x_1e_1\otimes e_2^2.$$

Hence we get $d_2(h) = -X_2^2g_1 + X_1^2g_2 + X_1X_2g_3$.

3. Stanley–Reisner rings with pure resolutions

In this section we want to describe the complex of Theorem 2.2 in case R is a polynomial ring, and M is a Stanley–Reisner ring $k[\Delta]\cong R/I_\Delta$. Then, due to Hochster [8], the Koszul homology may be interpreted as reduced simplicial cohomology of certain simplicial complexes. In fact,

$$H_i(\boldsymbol{x};k[\Delta])\cong \bigoplus_{W\subseteq V}\tilde{H}^{|W|-i-1}(\Delta_W;k). \tag{5}$$

Here $V=\{v_1,\dots,v_n\}$ is the vertex set of the simplicial complex Δ, and Δ_W is the simplicial complex restricted to W. i.e. the simplicial complex consisting of all faces $F\in\Delta$ whose vertices belong to W. Furthermore, $\tilde{H}^{\cdot}$ denotes reduced simplicial cohomology. We refer the reader to [2, 10] for terminology and some basic facts concerning Stanley–Reisner rings.

Let us describe Hochster's theorem in more details: One decomposes $K(\boldsymbol{x};k[\Delta])$ into a direct sum of subcomplexes

$$K(\boldsymbol{x};k[\Delta]) = D\oplus\left(\bigoplus_{W\subseteq V}K^W\right).$$

Here D is an exact subcomplex of $K(\boldsymbol{x};k[\Delta])$, and K^W is spanned as a k-vector space by the elements

$$x^F e_{W\setminus F},\quad F\in\Delta_W.$$

We assume that V is linearly ordered. If $F=\{v_{i_0},\dots,v_{i_k}\}$, $v_{i_0}<\cdots<v_{i_k}$, and $W\setminus F=\{v_{j_0},\dots,v_{j_l}\}$, $v_{j_0}<\cdots<v_{j_l}$, then $x^F=x_{i_0}\cdots x_{i_k}$ and $e_{W\setminus F}=e_{j_0}\wedge\cdots\wedge e_{j_l}$.

Let $\tilde{C}_{\boldsymbol{\cdot}}(\Delta_W;k)$ be the reduced chain complex of Δ_W with values in k. The homology of the dual complex $\tilde{C}^{\boldsymbol{\cdot}}(\Delta_W;k)=\operatorname{Hom}_k(\tilde{C}_{\boldsymbol{\cdot}}(\Delta_W;k),k)$ gives the simplicial cohomology. Recall that $\tilde{C}^j(\Delta_W;k)$ has the k-basis $\{F^*: F\in\Delta_W, |F|=j+1\}$, where $F^*:\tilde{C}_j(\Delta_W;k)\to k$ is the k-linear map with

$$F^*(G)=\begin{cases}1 & \text{if } G=F,\\ 0 & \text{if } G\neq F\end{cases}$$

for $G \in \Delta_W$, $|G| = j+1$. Moreover, the differential $\tilde{\partial}$ of $\tilde{C}^{\cdot}(\Delta_W;k)$ is given by

$$\tilde{\partial}(F^*) = \sum_r (-1)^{\alpha(F,v_r)}(F\cup\{v_r\})^*,$$

where the sum is taken over all r such that $v_r \in W\setminus F$ and $F\cup\{v_r\} \in \Delta_W$, and where $\alpha(F,v_r) = |\{v_i \in F : v_i < v_r\}|$.

Set

$$\gamma(F,W) = \sum_{v_r\in W\setminus F} \alpha(W,v_r),$$

and define the following isomorphisms of vector-spaces:

$$\Phi_i : \tilde{C}^{|W|-i-1}(\Delta_W;k) \to K_i^W, \qquad F^* \mapsto (-1)^{\gamma(F,W)} x^F e_{W\setminus F}.$$

Then for all i the diagrams

$$\begin{array}{ccc} \tilde{C}^{|W|-i-1}(\Delta_W;k) & \xrightarrow{\Phi_i} & K_i^W \\ \downarrow & & \downarrow \\ \tilde{C}^{|W|-i}(\Delta_W;k) & \xrightarrow{\Phi_{i-1}} & K_{i-1}^W \end{array}$$

commute. Hence Φ is an isomorphism of complexes. In particular, we obtain the isomorphisms (5).

From (5) we deduce that $k[\Delta]$ has a pure R-resolution of type $(a_1, a_2, \ldots)$ if and only if for all i and all $W \subseteq V$,

$$\tilde{H}^{|W|-i-1}(\Delta_W;k) = 0 \quad \text{if} \quad |W| \neq a_i.$$

In this case the resolution is of the form

$$\cdots \longrightarrow \bigoplus_W \tilde{H}^{a_i-i-1}(\Delta_W;k)\otimes_k R(-a_i)$$

$$\xrightarrow{\vartheta_i} \bigoplus_W \tilde{H}^{a_{i-1}-i}(\Delta_W;k)\otimes_k R(-a_{i-1}) \longrightarrow \cdots$$

In order to describe the differential ϑ in terms of simplicial cohomology we identify L (see the introduction) via Φ with the following complex (where we skipped the exact pieces coming from D)

$$\cdots \xrightarrow{\partial} \bigoplus_W \tilde{C}^{|W|-i-1}(\Delta_W;k)\otimes S \xrightarrow{\partial} \bigoplus_W \tilde{C}^{|W|-i}(\Delta_W;k)\otimes S \xrightarrow{\partial} \cdots$$

Here $\partial = \partial' + \partial''$ with $\partial' = \tilde{\partial}\otimes S$ and

$$\partial''(F^*\otimes a) = \sum_{v_r\in W\setminus F} d_r, \qquad d_r \in \tilde{C}^{|W|-i}(\Delta_{W\setminus\{v_r\}};k),$$

$$d_r = (-1)^{\alpha(W,v_r)+|W\setminus F|} F^*\otimes e_r a.$$

Now by Theorem 2.2 the differential ϑ_i may be described as follows:

Given $[c] \otimes 1 \in \tilde{H}^{a_i - i - 1}(\Delta_W; k) \otimes R_0$, one find elements

$$c_k \in \bigoplus_{|W| = a_i - k} \tilde{C}^{a_i - i - 1 - k}(\Delta_W; k) \otimes R_k, \quad k = 0, \dots, r_i = a_i - a_{i-1} - 1$$

such that

$$c_0 = c \quad \text{and} \quad \partial'' c_{s-1} + \partial' c_s = 0 \text{ for all } s = 1, \dots, r_i.$$

Write $\partial'' c_{r_i} = \sum_{|U| = a_{i-1}} c_U \otimes h_U$ with certain cycles $c_U \in \tilde{C}^{|U| - i - 1}(\Delta_U; k)$. Then

$$\vartheta_i([c] \otimes 1) = \sum_U [c_U] \otimes h_U.$$

As an application we describe the resolution of a one-dimensional simplicial complex Δ which has no cycles. Such a simplicial complex is called a *forest*, its connected components are called the *trees* (of the forest). Since Δ_W is again a forest for any subset W of the vertex set V, and since for any forest Γ one has $\tilde{H}^i(\Gamma; k) = 0$ for all $i > 0$, it follows at once from Hochster's formula (5) that Δ has a 2-linear resolution, i.e. a pure resolution of type $(2, 1, 1, \dots)$.

Let $v_{k_1}, \dots, v_{k_t}$ be the vertices of a tree T of Δ_W. Then $\sum_{j=1}^{t} \{v_{k_j}\}^*$ is a cycle of $\tilde{H}^0(\Delta_W; k)$ whose homology class we denote by $c(T)$.

Note that the k-vector-space $\tilde{H}^0(\Delta_W; k)$ is generated by the elements $c(T)$, T a tree of Δ_W. Furthermore, $\sum_T c(T) = 0$ is the only relation among the generators.

Let $v_r \in W$; then $\sum_{v_{k_j} \neq v_r} \{v_{k_j}\}^*$ is a cycle in $\tilde{H}^0(\Delta_{W \setminus \{v_r\}}; k)$. We denote its homology class by $c(T)_r$. The restriction $T_{W \setminus \{v_r\}}$ of the tree T to $W \setminus \{v_r\}$ is a union of trees $T^{(s)}$, and $c(T)_r = \sum_s c(T^{(s)})$ in $\tilde{H}^0(\Delta_{W \setminus \{v_r\}}; k)$.

As the resolution of $k[\Delta]$ is 2-linear, the differential ϑ_i is linear for $i > 1$, and thus is induced by ∂''. Hence for $c(T) \in \tilde{H}^0(\Delta_W; k), |W| = i + 1$, we get

$$\vartheta_i(c(T) \otimes 1) = \sum_r d_r, \quad d_r \in \tilde{H}^0(\Delta_{W \setminus \{v_r\}}; k) \otimes S, \; d_r = (-1)^{\alpha(W, v_r) + i} c(T)_r \otimes e_r.$$

Simplifying notation we let $R(W)$ be the free R-module generated by the trees T of Δ_W, and with the only relation $\sum T = 0$ among the generators. If the restriction of T to $\Delta_{W \setminus \{v_r\}}$ is the union of the trees $T^{(s)}$, we set $T_r = \sum_s T^{(s)}$ in $R(\Delta_{W \setminus \{v_r\}})$.

Proposition 3.1. *Let Δ be a forest. Then $k[\Delta]$ has the following R-free resolution:*

$$\cdots \xrightarrow{\vartheta_3} \bigoplus_{|W| = 3} R(W)(-3) \xrightarrow{\vartheta_2} \bigoplus_{|W| = 2} R(W)(-2) \xrightarrow{\vartheta_1} R \longrightarrow 0.$$

If $W = \{v_i, v_j\}$, $v_i < v_j$ one has $R(W) = 0$ for $\{v_i, v_j\} \in \Delta$, and $R(W) = R\{v_i\}^* = R\{v_j\}^*$ for $\{v_i, v_j\} \notin \Delta$. In the latter case, $\vartheta_1 \{v_i\}^* = -\vartheta_1 \{v_j\}^* = X_i X_j$. If $|W| = i + 1 > 2$, let T be one of the trees generating $R(W)$. Then

$$\vartheta_i(T) = \sum_r d_r, \quad d_r \in R(W \setminus \{v_r\}), \; d_r = \sum_r (-1)^{\alpha(W, v_r)} X_r T_r.$$

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